

**Higher-Dimensional Integrable Systems  
from  
Multilinear Evolution Equations**

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**Abstract**

A multilinear  $M$ -dimensional generalization of Lax pairs is introduced and its explicit form is given for the recently discovered class of time-harmonic, integrable, hypersurface motions in  $\mathbb{R}^{M+1}$ .

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In [1] the explicit form of a triple  $(L, M_1, M_2)$ , depending on 2 spectral parameters and 4 time-dependent functions  $x_i(t, \varphi^1, \varphi^2, \varphi^3)$  from a 3-dimensional Riemannian manifold  $\Sigma$  to  $\mathbb{R}$  was given such that (with  $\rho$  a non-dynamical density on  $\Sigma$ )

$$\dot{L} = \frac{1}{\rho} \in_{rsu} \frac{\partial L}{\partial \varphi^u} \frac{\partial M_1}{\partial \varphi^r} \frac{\partial M_2}{\partial \varphi^s} \quad (1)$$

is equivalent to the equations

$$\dot{x}_i = \frac{1}{\rho} \in_{ii_1i_2i_3} \in_{r_1r_2r_3} \partial_{r_1} x_{i_1} \partial_{r_2} x_{i_2} \partial_{r_3} x_{i_3}, \quad (2)$$

describing the integrable motion of a hypersurface  $\hat{\Sigma}$  in  $\mathbb{R}^4$  whose time-function (the time at which  $\hat{\Sigma}$  reaches a point  $\mathbf{x} \in \mathbb{R}^4$ ) is harmonic [2].

The purpose of this note is to give the explicit generalization of this construction to an arbitrary number of dimensions,  $M (= \dim \Sigma)$ . Let

$$z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4, \dots \quad . \quad (3)$$

For **even**  $M$  ( $= 2m$ ) one may take

$$\begin{aligned} L &= \sum_{a=1}^m \left( \lambda_a z_a - \frac{\bar{z}_a}{\lambda_a} \right) + 2\sqrt{m} x_N \\ M_a &= \frac{i}{2} \left( \lambda_a z_a + \frac{x_N}{\sqrt{m}} \right) \quad a = 1 \dots m \\ M_{m+a'} &= \left( \frac{4}{m} \right)^{\frac{1}{N-3}} \left( \frac{\bar{z}_{m+1-a'}}{\lambda_{m+1-a'}} - \frac{\bar{z}_{m-a'}}{\lambda_{m-a'}} \right) \quad a' = 1 \dots m-1 \end{aligned} \quad (4)$$

depending on  $m$  spectral parameters,  $\lambda_a$ , and  $N = M+1$  functions  $x_i(t, \varphi^1, \dots, \varphi^M)$ ; letting

$$\{f_1, \dots, f_M\} := \frac{1}{\rho(\varphi^1 \dots \varphi^M)} \in_{r_1 \dots r_M} \partial_{r_1} f_1 \dots \partial_{r_M} f_M, \quad (5)$$

$$\dot{L} = \{L, M_1, M_2, \dots, M_{2m-1}\} \quad (6)$$

will then be equivalent to the equations of motion (as above,  $\bar{z}_a$  denoting the complex conjugate of  $z_a$ )

$$\begin{aligned} \dot{z}_a &= -i \left( \frac{i}{2} \right)^{m-1} \{z_a, z_{a+1}, \bar{z}_{a+1}, \dots, z_{a-1}, \bar{z}_{a-1}\} \\ \dot{x}_N &= \left( \frac{i}{2} \right)^m \{z_1, \bar{z}_1, \dots, z_m, \bar{z}_m\}. \end{aligned} \quad (7)$$

For **odd**  $M$  ( $= 2m + 1$ ), rather than given a particular form of  $L, M_1, \dots, M_{2m}$  that would make

$$\dot{L} = \{L, M_1, \dots, M_{2m}\} \quad (8)$$

equivalent to the equations of motion

$$\dot{z}_\alpha = -i \left( \frac{i}{2} \right)^m \{z_\alpha, z_{\alpha+1}, \bar{z}_{\alpha+1}, \dots, z_{\alpha-1}, \bar{z}_{\alpha-1}\}, \quad \alpha = 1, \dots, n = m + 1 \quad , \quad (9)$$

let me in this case stress the simple general nature of the construction: Think of

$$L = L_1 \lambda_1 z_1 + L_2 \frac{\bar{z}_1}{\lambda_1} + \dots + L_{N-1} \lambda_m z_m + L_N \frac{\bar{z}_m}{\lambda_m} \quad , \quad (10)$$

and likewise  $M_1, \dots, M_{2m}$ , as  $N = 2n$  dimensional vectors  $\mathbf{L}, \mathbf{M}_1, \dots, \mathbf{M}_{2m}$  in a vectorspace  $V$  with basis  $\lambda_1 z_1, \dots, \frac{\bar{z}_m}{\lambda_m}$ . The wanted equivalence of (8) with (9) may then be stated as the requirement that

$$\det(\mathbf{L}, \mathbf{M}_1 \mathbf{M}_2 \dots \mathbf{M}_{2m} \mathbf{e}_j) = i \left( \frac{2}{i} \right)^m \hat{\mathbf{L}} \cdot \mathbf{e}_j \quad , \quad (11)$$

where  $\mathbf{e}_j = (0 \dots 010 \dots 0)^{tr}$  and

$$\hat{\mathbf{L}} = (L_2, L_1, L_4, L_3, \dots, L_N, L_{N-1}) \quad . \quad (12)$$

Multiplying (11) with the  $j$ -th component of  $\mathbf{L}$  (or any of the  $\mathbf{M}$ 's), and summing over  $j$ , one finds that all  $2m + 1$  vectors  $\mathbf{L}, \mathbf{M}_1, \dots, \mathbf{M}_{2m}$  have to be perpendicular to  $\hat{\mathbf{L}}$ ; in particular,

$$\hat{\mathbf{L}} \cdot \mathbf{L} = 2(L_1 L_2 + \dots + L_{N-1} L_N) = 0 \quad . \quad (13)$$

Choosing  $\mathbf{M}_1, \dots, \mathbf{M}_{2m}$  to be also perpendicular to  $\mathbf{L}$ , the only remaining condition, obtained by multiplying (11) by  $\hat{L}_j$  (and summing), becomes ( $\sim$  denoting the projection onto the  $2n - 2 = 2m$ -dimensional orthogonal complement of the  $\mathbf{L}, \hat{\mathbf{L}}$ -plane)

$$\det(\tilde{\mathbf{M}}_1, \dots, \tilde{\mathbf{M}}_{2m}) = i \left( \frac{2}{i} \right)^m \quad , \quad (14)$$

which exhibits the large freedom in choosing the  $\mathbf{M}$ 's (for fixed  $\mathbf{L}$ ). A similar reasoning applies directly to the real equations (cp. [2]),

$$\dot{x}_i = \frac{1}{M!} \in_{ii_1 \dots i_M} \{x_{i_1}, \dots, x_{i_M}\}; \quad (15)$$

the Ansatz  $L = \sum_{i=1}^N \mathbb{L}_i x_i, M_1 = \sum \mathbb{M}_{1i} x_i, \dots$  immediately implies

$$\sum_{i=1}^N \mathbb{L}_i^2 = 0, \quad (16)$$

making  $L^l, l \in \mathbb{N}$ , a harmonic polynomial of  $x_1, \dots, x_N$  (while its integral is time-independent, due to (6) resp. (8)) no matter whether  $m$  is odd, or even.

## REFERENCES

- [1] J. Hoppe; ‘On  $M$ -Algebras, the Quantization of Nambu Mechanics, and Volume Preserving Diffeomorphisms’, ETH-TH/95-33 (preprint)
- [2] M. Bordemann, J. Hoppe; ‘Diffeomorphism Invariant Integrable Field Theories and Hypersurface Motions in Riemannian Manifolds’, ETH-TH/95-31, FR-THEP-95-26 (preprint)